Ark. Mat. (2007)
DOI: 10.1007/s11512-006-0039-0

Persistence of Anderson localization in Schrödinger operators with decaying random potentials

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Abstract. We show persistence of both Anderson and dynamical localization in Schrödinger operators with non-positive (attractive) random decaying potential. We consider an Anderson-type Schrödinger operator with a non-positive ergodic random potential, and multiply the random potential by a decaying envelope function. If the envelope function decays slower than $|x|^{-2}$ at infinity, we prove that the operator has infinitely many eigenvalues below zero. For envelopes decaying as $|x|^{-\alpha}$ at infinity, we determine the number of bound states below a given energy $E<0$, asymptotically as $\alpha\downarrow 0$. To show that bound states located at the bottom of the spectrum are related to the phenomenon of Anderson localization in the corresponding ergodic model, we prove: (a) these states are exponentially localized with a localization length that is uniform in the decay exponent α; (b) dynamical localization holds uniformly in α.

1. Introduction and results

A mathematical proof of the existence of absolutely continuous (or just continuous) spectrum for a multidimensional Schrödinger operator with random ergodic potential is still a challenge. Up to date there is no proof of *any* continuous spectrum for ergodic random Schrödinger operators in d-dimensional spaces, neither on the lattice nor in the continuum. The only known result is the existence of absolutely continuous spectrum for the Anderson model on the Bethe lattice [Kl] (see also [ASW] and [FHS]). The only proof of existence of a localization-delocalization transition in finite dimensions for a typical ergodic random Schrödinger operator

A. Figotin was sponsored by the Air Force Office of Scientific Research Grant FA9550-04-1-0359.

A. Klein was supported in part by NSF Grant DMS-0457474.

P. Müller was supported in part by the Deutsche Forschungsgemeinschaft under Grant Mu 1056/2–1.

is for random Landau Hamiltonians ($d=2$), where non-trivial transport has been shown to occur near each Landau level [GKS]. (See [JSS] for a special delocalization phenomenon in one-dimensional random polymer models.)

To gain insight into this fundamental question, one may impose a decaying envelope on the ergodic random potential, and study the absolutely continuous spectrum for the new Schrödinger operator with random decaying potential as a step towards the understanding the original problem, see [Kr], [KKO], [B1], [B2], [RoS], [De], [BoSS] and [Ch]. Relaxing the decay conditions, one hopes to get an idea of the nature of the continuous spectrum for the original ergodic random Schrödinger operator. If the imposed envelope decays fast enough, regular scattering theory applies, and one may conclude that the spectrum is absolutely continuous regardless of the randomness. This indicates that the essence of the original problem is to establish the existence of continuous spectrum "in spite of the randomness" of the ergodic potential. Since randomness leads to Anderson localization and the existence of non-trivial pure point spectrum, one must answer the question of when continuous spectrum can coexist with Anderson localization. In particular, we may ask if this coexistence phenomenon can already be seen in Schrödinger operators with random decaying potential.

In this paper we show persistence of both Anderson and dynamical localization in Schrödinger operators with non-positive (attractive) random decaying potential. We consider the random Schrödinger operator

$$H_{\alpha,\lambda,\omega} := -\Delta + \lambda\gamma_\alpha V_\omega \quad \text{on } \mathrm{L}^2(\mathbb{R}^d), \tag{1}$$

where $\lambda > 0$ is the disorder parameter, $\alpha \geq 0$, γ_α is the *envelope function*

$$\gamma_\alpha(x) := \langle x \rangle^{-\alpha}, \quad \text{where } \langle x \rangle := \sqrt{1+|x|^2}, \tag{2}$$

and V_ω is the *non-positive* random potential given by

$$V_\omega(x) := -\sum_{j\in\mathbb{Z}^d} \omega_j \, u(x-j). \tag{3}$$

Here $\{\omega_j\}_{j\in\mathbb{Z}^d}$ are independent identically distributed random variables on some probability space $(\Omega, \mathbb{P})$, with $0 \leq \omega_0 \leq 1$ and $\mathbb{E}[\omega_0] > 0$. The single-site potential $u \in \mathrm{L}^\infty(\mathbb{R}^d)$ is assumed to satisfy

$$0 \leq u \leq u_0, \quad \operatorname{supp} u \text{ is compact} \quad \text{and} \quad v := \int_{\mathbb{R}^d} u(x)\, dx > 0, \tag{4}$$

with $u_0 > 0$ a constant. We note that the support of u may be arbitrarily small. Under these hypotheses $H_{\alpha,\lambda,\omega}$ is self-adjoint on the domain of the Laplacian Δ for every $\omega \in \Omega$.

In the special case of a constant envelope function, obtained by setting $\alpha=0$ in (2), $H_{\lambda,\omega}:=H_{0,\lambda,\omega}$ is the usual Anderson-type Schrödinger operator with a non-positive ergodic random potential. Due to ergodicity, the spectrum $\sigma(H_{\lambda,\omega})$ of $H_{\lambda,\omega}$, as well as the spectral components in the Lebesgue decomposition, do not depend on ω for $\mathbb{P}$-almost every $\omega\in\Omega$, see [CL] and [PF]. If the single-site distribution $\mathbb{P}(\omega_0\in\cdot)$ has a bounded Lebesgue density, it is also well-known that $H_{\lambda,\omega}$ exhibits Anderson localization, both spectral and dynamical, in a neighborhood above the non-random bottom $E_0(\lambda)<0$ of its spectrum, see [CoH], [GK3] and [S]. The latter is also true if ω_0 is a Bernoulli random variable ($\mathbb{P}(\omega_0=0)=\mathbb{P}(\omega_0=1)=\frac{1}{2}$), and may be shown by modifying [BK] as in [GHK2]. For non-ergodic random Schrödinger operators, like $H_{\alpha,\lambda,\omega}$ with $\alpha>0$, one cannot expect non-randomness of the spectrum and of the spectral components in general.

If $\alpha>1$, one is able to construct wave operators for $H_{\alpha,\lambda,\omega}$. This implies that for all $\lambda>0$ the absolutely continuous spectrum of $H_{\alpha,\lambda,\omega}$ coincides with $[0,\infty[$ for $\mathbb{P}$-almost all $\omega\in\Omega$ [HK] (see also [Kr]). Recent results suggest that this should be true for all $\alpha>\frac{1}{2}$, see [B1], [B2] and [De]. Of course, the primary interest in models like (1) is for small parameters α, when they are "close" to the ergodic random Schrödinger operator $H_{\lambda,\omega}$.

Since the random potential in (3) is non-positive, $H_{\alpha,\lambda,\omega}$ can only have discrete spectrum at energies below zero: its essential spectrum is almost surely equal to $[0,\infty[$, cf. [CL, Theorem II.4.3] or [RS, Example 6 in Section 13.4]. Consequently, for any given $\alpha>0$ the random operator $H_{\alpha,\lambda,\omega}$ exhibits localization of eigenfunctions and even dynamical localization in any given interval below zero for $\mathbb{P}$-almost every $\omega\in\Omega$. But does this localization regime have anything in common with the well-studied region of complete localization—as it was called in [GK4] and [GKS]—that occurs for $\alpha=0$, i.e., for the corresponding ergodic random Schrödinger operator $H_{\lambda,\omega}$?

To get insight into this question, suppose a bound state of $H_{\alpha,\lambda,\omega}$ with energy $E<0$ was localized solely because of the presence of the envelope. Then it would be localized in a ball of size $|E|^{-1/\alpha}$, roughly. Outside this ball, in the classically forbidden region, it would decay exponentially fast. Hence, the slower the decay of the envelope, the weaker this type of localization would be. In particular, it would disappear in the limit as $\alpha\downarrow 0$. Our main result, given in part (3) of Theorem 1, shows that this is not the case. Localization occurs uniformly in α so that bound states of $H_{\alpha,\lambda,\omega}$ are localized because of the presence of randomness, and not because of the decaying envelope. Likewise, dynamical localization holds uniformly in $\alpha\geq 0$. This is not a trivial property either, because we show in part (2) of Theorem 1 that the number of contributing eigenfunctions diverges as $\alpha\downarrow 0$. Thus, Anderson localization persists also from a dynamical point of view.

In the formulation of Theorem 1, we use the notation

$$n(A,E) := \#\{E_j \le E : E_j \text{ is an eigenvalue of } A\} \tag{5}$$

for the number of eigenvalues of a self-adjoint operator A which do not exceed a given $E\in\mathbb{R}$—counted according to their multiplicities. (This number is always finite if A is bounded from below and has only discrete spectrum up to E.)

Theorem 1. *Let $H_{\alpha,\lambda,\omega}$ be as in* (1)–(4).

(1) *If $\alpha\in]0,2[$, then $H_{\alpha,\lambda,\omega}$ has infinitely many eigenvalues in $]-\infty,0[$ for $\mathbb{P}$-almost every $\omega\in\Omega$.*

(2) *Let $E_0(\lambda)<0$ denote the non-random bottom of the spectrum of $H_{\lambda,\omega}$. For $\mathbb{P}$-almost every $\omega\in\Omega$, the inequalities*

$$\begin{aligned} d\log\Big(\frac{1}{\nu_0(\lambda,E)}\Big) \le \liminf_{\alpha\downarrow 0}\,[\alpha\log n(H_{\alpha,\lambda,\omega},E)] &\le \limsup_{\alpha\downarrow 0}\,[\alpha\log n(H_{\alpha,\lambda,\omega},E)] \\ &\le d\log\Big(\frac{\lambda U_0}{|E|}\Big) \end{aligned} \tag{6}$$

hold for all $E\in]E_0(\lambda),0[$, where

$$\nu_0(\lambda,E) := \inf\{\nu\in]0,1[: E_0(\nu\lambda)<E\},$$
$$U_0 := \|\sum_{j\in\mathbb{Z}^d} u(\,\cdot-j)\|_\infty.$$

(3) *If the single-site distribution $\mathbb{P}(\omega_0\in\cdot\,)$ has a bounded Lebesgue density, then there exists an energy $E_1(\lambda)\in]E_0(\lambda),0[$ such that the following holds:*

(a) *For $\mathbb{P}$-almost every $\omega\in\Omega$, any eigenfunction $\varphi_{n,\alpha,\lambda,\omega}$ of $H_{\alpha,\lambda,\omega}$ with eigenvalue in $I_\lambda:=[E_0(\lambda),E_1(\lambda)]$ decays exponentially fast with a mass $m>0$. The mass m can be chosen independently of $\alpha\ge 0$. More precisely, one has the following SULE([1])-like property: there exists a localization center $x_{n,\alpha,\lambda,\omega}$ located in the ball centered at the origin and of radius $\mathcal{O}(|E|^{-1/\alpha})$, if $|E|<2\lambda u_0$ and $\alpha\le 1$, and $\mathcal{O}(|E|^{-1})$ otherwise, such that for any $\varepsilon>0$,*

$$\|\chi_x\varphi_{n,\alpha,\lambda,\omega}\| \le C_{\varepsilon,\lambda,\omega}\, e^{|x_{n,\alpha,\lambda,\omega}|^\varepsilon}\, e^{-m|x-x_{n,\alpha,\lambda,\omega}|} \tag{7}$$

for all $x\in\mathbb{R}^d$ and $\alpha\ge 0$, where $C_{\varepsilon,\lambda,\omega}>0$ is a constant independent of α and χ_x is the indicator (characteristic) function of the unit cube in $\mathbb{R}^d$ centered at x;

(b) *One has uniform dynamical localization: for any $p\ge 0$,*

$$\sup_{\alpha\ge 0}\sup_{|f|\le 1}\mathbb{E}[\|\langle x\rangle^p f(H_{\alpha,\lambda,\omega})\chi_{I_\lambda}(H_{\alpha,\lambda,\omega})\chi_0\|_2^2]<\infty, \tag{8}$$

([1]) Semi-uniformly localization of eigenfunctions *where* $\|\cdot\|_2$ *stands for the Hilbert–Schmidt norm and the supremum is taken over all measurable functions* $f:\mathbb{R}\to\mathbb{R}$ *which are bounded by one.*

Remark 1. For $\alpha\in]1,2[$ the operator $H_{\alpha,\lambda,\omega}$ has both absolutely continuous spectrum for energies in $[0,\infty[$ (see the discussion above) and infinitely many eigenvalues below zero.

Remark 2. It follows from the proof, of part (3) of Theorem 1 (see Section 4) that the interval I_λ corresponds to the range of energies where one can prove localization for $H_{\lambda,\omega}$, that is, energies for which the initial-scale estimate of the multiscale analysis can be established for the corresponding ergodic operator. The rate of exponential decay also coincides with the one of the ergodic model. In other words, the eigenfunctions have the same localization length uniformly in α.

Remark 3. In a few typical cases, one can show that the length of the interval I_λ scales (at least) like λ. At small disorder λ, this is proved in [W], [Klo2] and [Klo3]. At large disorder, this is shown in [GK2] and [GK3] under the assumption that the single-site potential u satisfies the covering condition $u\geq v_0\chi_{\Lambda_1}>0$ for some $v_0>0$. Such an assumption can be removed using an averaging procedure as in [BK] and [GHK1], in which case it is enough to assume that there exists $\delta>0$ and $v_0>0$ such that $u\geq v_0\chi_{\Lambda_\delta}$, but the length of the interval I_λ then scales as λ^ρ for some $\rho\in]0,1[$.

Remark 4. We point out that our result does not apply to positive potentials, i.e., with a reversed sign in (3). For instance, the standard proof of the initial-scale estimate would fail for boxes that are far from the origin. In fact, at least for α large enough, one expects the absolutely continuous spectrum to fill up the entire positive half-line. The existence of a localized phase for low energies and small α is an open problem in this case, see [B1].

Remark 5. Dynamical localization is just one property of the region of complete localization, further properties can be found in [GK2] and [GK4]. In particular, following [GK4], one can show decay of the kernel of the Fermi projector and strong uniform decay of eigenfunction correlations (SUDEC), uniformly in α. We would like to emphasize that while for ergodic models these properties are known to be characterizations of the region of complete localization, i.e., they provide necessary and sufficient conditions, adding the envelope destroys the equivalence, and only the “necessary part” survives.

This paper is organized as follows. In Sections 2, 3, and 4, we prove parts (1), (2), and (3), respectively, of Theorem 1.

2. Infinitely many bound states

In this section we deduce part (1) of Theorem 1 from a corresponding result for slightly more general Schrödinger operators with decaying random potentials. Given any non-negative function $0 \le \gamma \in \mathrm{L}^\infty(\mathbb{R}^d)$, we consider the random Schrödinger operator

$$H_\omega(\gamma) := -\Delta + \gamma V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d), \tag{9}$$

where V_ω is as in (3). For (9) to represent a Schrödinger operator with decaying randomness, we require that the envelope function γ vanishes at infinity, $\lim_{|x|\to\infty} \gamma(x) = 0$. Part (1) of Theorem 1 then follows immediately from the following result.

Theorem 2. *Let $H_\omega(\gamma)$ be as in* (9)*. Suppose that*

$$\gamma(x)|x|^2 \ge F(|x|) \quad \textit{for all } |x| > R_0, \tag{10}$$

where $R_0 > 0$ and $F\colon [R_0, \infty[\to]0, \infty[$ is a strictly increasing function such that

$$\lim_{r\to\infty} F(r) = \infty \quad \textit{and} \quad \lim_{r\to\infty} \frac{F(r)}{r^2} = 0. \tag{11}$$

Then $H_\omega(\gamma)$ has infinitely many eigenvalues in $]-\infty, 0[$ for $\mathbb{P}$-almost every $\omega \in \Omega$.

Remark 6. Theorem 2 extends known deterministic results, namely those in [RS, Theorem 13.6] and [DHS, Theorem A.3(iii) and (iv)], to the random case. Thanks to randomness we do not have to require that each realization V_ω of the potential stays away from zero, whereas this has to be assumed for the deterministic potentials in [RS] and [DHS]. Note also [DHKS, Theorem 5.3(ii)], which gives an infinite number of eigenvalues for discrete random Schrödinger operators on $\mathbb{Z}_+$ with an arbitrary (deterministic) potential subject to $\limsup_{n\to\infty} |V(n)| n^{1/2} > 2$.

Remark 7. Theorem 2 is almost optimal, as [DHS, Theorem A.3(i) and (ii)] (see also [RS, Theorem 13.6]) implies that $H_\omega(\gamma)$ has at most finitely many eigenvalues in $]-\infty, 0[$ for $\mathbb{P}$-almost every $\omega \in \Omega$, whenever $\gamma(x)\,|x|^2 \le (1-(d/2))^2$ for all $|x| > R_0$, if $d \ge 3$, or whenever $\gamma(x)\,|x|^2 \le (2\log|x|)^{-2}$ for all $|x| > R_0$, if $d = 2$.

Proof of Theorem 2. Without loss of generality, we shall assume that the support of u is included in Λ_1, where $\Lambda_L :=]L/2, L/2[^d$ for $L > 0$ (the smaller $\operatorname{supp} u$, the smaller is the number of eigenvalues). Also, on account of (10), we may assume that F grows to infinity as slowly as we want (the smaller F, the smaller the number of eigenvalues). We shall prove that for all $L \ge L_0 > 2R_0$, with L_0 large enough and depending on $d, v, R_0, \mathbb{E}[\omega_0]$, and F, there is a constant $c>0$ such that

$$\mathbb{P}(A_L) \geq 1 - [F(L)]^{d/4} \mathrm{e}^{-cL^d [F(L)]^{-d/4}}, \tag{12}$$

where

$$A_L := \{\omega \in \Omega : H_\omega(\gamma) \text{ has at least } \varkappa F(L)^{d/4} \text{ eigenvalues in }]-\infty, 0[\}, \tag{13}$$

with $0<\varkappa<1$ being some constant depending only on d and R_0. The theorem then follows by taking F with a growth rate that is slow enough and using the Borel–Cantelli lemma.

To prove (12), we set

$$\ell := [F(L)]^{-1/4} L, \tag{14}$$

and divide the shell $\Lambda_L \setminus \Lambda_{2R_0}$ into $N = \mathcal{O}[(L/\ell)^d - (2R_0/\ell)^d]$ non-overlapping cubes $\Lambda_\ell(n)$, $n=1, ..., N$, of side length ℓ. (More precisely, we only consider the cubes contained in the shell.) Clearly, we have $\varkappa [F(L)]^{d/4} \leq N \leq [F(L)]^{d/4}$ with some $\varkappa$ as above. For each $n=1, ..., N$ and each $\omega \in \Omega$ there exists a function $\varphi_n \in \mathrm{dom}(H_\omega(\gamma)) = \mathrm{dom}(-\Delta)$ such that

(1) $\|\varphi_n\| = 1$;
(2) φ_n has compact support in $\Lambda_\ell(n)$;
(3) $\varphi_n|_{\Lambda_\ell^{\mathrm{int}}(n)} = c_0 \ell^{-d/2}$, where $\Lambda_\ell^{\mathrm{int}}(n) := \{x \in \Lambda_\ell(n) : \mathrm{dist}_\infty(x, \partial\Lambda_\ell(n)) \geq \ell/4\}$;
(4) $\big\| |\nabla \varphi_n| \big\|_\infty \leq c_0 \ell^{-1-d/2}$;

where the constant $c_0>0$ depends only on the dimension, and the distance in (3) is measured with respect to the maximum norm in $\mathbb{R}^d$. Note that the φ_n's have disjoint supports, and hence are mutually orthogonal. From the above, and since $\Lambda_L \setminus \Lambda_{2R_0}$ is contained in the annulus with $|x| > R_0$ and $|x| \leq \sqrt{d} L/2$, we conclude for every $n=1, ..., N$ and every $\omega \in \Omega$ that

$$\begin{aligned}
\langle \varphi_n, H_\omega(\gamma)\varphi_n \rangle &\leq \langle \varphi_n, (-\Delta + \widetilde{F}(L) L^{-2} V_\omega)\varphi_n \rangle \\
&\leq \|\nabla \varphi_n\|^2 - \widetilde{F}(L) L^{-2} \sum_{i \in \mathbb{Z}^d : \Lambda_1(i) \subset \Lambda_\ell^{\mathrm{int}}(n)} \omega_i \langle \varphi_n, u(\,\cdot - i)\varphi_n \rangle \\
&\leq c_0 \ell^{-2} - \widetilde{F}(L) L^{-2} c_0^2 \ell^{-d} v \sum_{i \in \mathbb{Z}^d : \Lambda_1(i) \subset \Lambda_\ell^{\mathrm{int}}(n)} \omega_i,
\end{aligned} \tag{15}$$

where $\widetilde{F}(L) := (4/d) F\big(\sqrt{d} L/2\big)$. Recalling (4) and the monotonicity of F, we infer the existence of a constant $c_1>0$ such that

$$\langle \varphi_n, H_\omega(\gamma)\varphi_n \rangle \leq c_0 \ell^{-2} - c_1 \ell^{-2} [\widetilde{F}(L)]^{1/2} v X_n^{(\omega)}(\ell), \tag{16}$$

where

$$X_n^{(\omega)}(\ell) := \frac{1}{Z_n(\ell)} \sum_{i\in\mathbb{Z}^d:\Lambda_1(i)\subset\Lambda_\ell^{\mathrm{int}}(n)} \omega_i \tag{17}$$

and $Z_n(\ell)$ is the number of terms in the i-sum in (17). Now, pick $0<\mu<\mathbb{E}[\omega_0]$. By a large-deviation estimate, there exists a constant $c_2>0$ such that

$$\mathbb{P}(X_n^{(\omega)}(\ell) \geq \mu \text{ for all } n=1,...,N) \geq 1-\sum_{n=1}^{N} \mathbb{P}(X_n(\ell) \leq \mu) \geq 1-N\mathrm{e}^{-c_2\ell^d} \tag{18}$$

holds for all sufficiently large ℓ. Thus, it follows from (16) and (18) that for L large enough ensuring $c_1[\widetilde{F}(L)]^{1/2}v\mu>c_0$, we have

$$\mathbb{P}\Big(\max_{n=1,\ldots,N}\langle\varphi_n, H_\omega(\gamma)\varphi_n\rangle < 0\Big) \geq 1-[F(L)]^{d/4}\mathrm{e}^{-c_2\ell^d}. \tag{19}$$

The bound (12) now follows from (19) and the min-max principle. Indeed, we have the representation

$$\lambda_N^{(\omega)} = \inf_{\mathcal{V}_N\subset\mathrm{L}^2(\mathbb{R}^d)} \sup_{\psi\in\mathcal{V}_N:\|\psi\|=1} \langle\psi, H_\omega(\gamma)\psi\rangle\,, \tag{20}$$

for the Nth eigenvalue $\lambda_N^{(\omega)}$ (counted from the bottom of the spectrum and including multiplicities) of $H_\omega(\gamma)$. The infimum in (20) is taken over all N-dimensional subspaces $\mathcal{V}_N$ of Hilbert space. Therefore we have

$$\lambda_N^{(\omega)} \leq \sup_{\psi\in\mathrm{span}\{\varphi_1,\ldots,\varphi_N\}:\,\|\psi\|=1} \langle\psi, H_\omega(\gamma)\psi\rangle \max_{n=1,\ldots,N}\langle\varphi_n, H_\omega(\gamma)\varphi_n\rangle < 0, \tag{21}$$

where we used the orthonormality of the φ_n's, their disjoint supports, and the locality of $H_\omega(\gamma)$. □

3. Counting the bound states

In this section we deduce part (2) of Theorem 1 from upper and lower bounds on the number of eigenvalues of $H_{\alpha,\lambda,\omega}$ in terms of the (self-averaging) integrated density of states N_λ of the ergodic random Schrödinger operator $H_{\lambda,\omega}$.

We recall from [CL] and [PF] that there exists a set $\Omega_0\subset\Omega$ of full probability, $\mathbb{P}(\Omega_0)=1$, such that the macroscopic limit

$$N_\lambda(E) := \lim_{L\to\infty} \frac{n\big(H_{\lambda,\omega}^{(L,X)}, E\big)}{L^d} \tag{22}$$

can be used to define a non-random, right-continuous and non-decreasing function N_λ on $\mathbb{R}$ in the sense that (22) holds for all $\omega \in \Omega_0$ and all continuity points $E \in \mathbb{R}$ of N_λ. The operator $H_{\lambda,\omega}^{(L,X)}$ in (22) denotes the restriction of $H_{\lambda,\omega}$ to the cube Λ_L. The boundary condition X can be arbitrary, as long as it renders the restricted operator self-adjoint. In particular Dirichlet (D) or Neumann (N) boundary conditions are allowed and lead to the same integrated density of states N_λ.

Since the envelope provides kind of an effective confinement for bound states of $H_{\alpha,\lambda,\omega}$ with energy below zero, one would expect that

$$\lim_{\alpha\downarrow 0} \frac{n(H_{\alpha,\lambda,\omega}, E)}{l_{\alpha,\lambda,E}^d} = N_\lambda(E) \tag{23}$$

for all $E \in]E_0(\lambda), 0[$ and for $\mathbb{P}$-almost all $\omega \in \Omega$, where $l_{\alpha,\lambda,E}$ is some effective, non-random "confinement length". We can prove the following result.

Theorem 3. *Let $H_{\alpha,\lambda,\omega}$ be as in* (1)–(4)*. Fix $\lambda > 0$. For all $\delta > 0$, $\nu \in]0,1[$, $\omega \in \Omega_0$ and $E < 0$ there exists $\alpha_0 > 0$ such that for all $\alpha \in]0, \alpha_0[$ we have*

$$\left(\frac{\nu^{-2/\alpha} - 1}{d}\right)^{d/2} \big(N_{\nu\lambda}(E) - \delta\big) \leq \frac{n(H_{\alpha,\lambda,\omega}, E)}{2^d} \leq \left(\frac{\lambda U_0}{|E|}\right)^{d/\alpha} \big(N_\lambda(E) + \delta\big). \tag{24}$$

The constant U_0 was defined in part (2) *of Theorem* 1*. In particular, the estimates in* (6) *hold for all $E \in]E_0(\lambda), 0[$ and all $\omega \in \Omega_0$.*

Corollary 1. *If, in addition, the single-site potential u is chosen such that $E_0(\lambda) = -\lambda U_0$ (e.g. if u is proportional to the characteristic function of the open unit cube around the origin, $u = U_0 \chi_{\Lambda_1}$), then we have $\nu_0(\lambda, E) = |E|/(\lambda U_0)$, and* (6) *implies the asymptotics*

$$\lim_{\alpha\downarrow 0} \left[\alpha \log n(H_{\alpha,\lambda,\omega}, E)\right] = d \log\left(\frac{\lambda U_0}{|E|}\right). \tag{25}$$

Remark 8. If the limit (23) exists in the situation of Corollary 1, then the confinement length obeys $\lim_{\alpha\downarrow 0}\left[\alpha \log l_{\alpha,\lambda,E}\right] = \log(\lambda U_0/|E|)$.

Proof of Theorem 3. First, we turn to the lower bound in (24). Let $\alpha > 0$, $\nu \in]0,1[$ and set

$$L_\alpha(\nu) := \frac{2}{\sqrt{d}} \left(\nu^{-2/\alpha} - 1\right)^{1/2}. \tag{26}$$

We observe that for every $\lambda > 0$ and $\omega \in \Omega_0$, the (non-decaying) random potential of the Dirichlet restriction $H_{0,\nu\lambda,\omega}^{(L_\alpha(\nu),D)}$ is bounded from below by the decaying random potential of the Dirichlet restriction $H^{(L_\alpha(\nu),D)}_{\alpha,\lambda,\omega}$. This implies

$$n(H_{\alpha,\lambda,\omega},E)\geq n\big(H^{(L_\alpha(\nu),D)}_{\alpha,\lambda,\omega},E\big)\geq n\big(H^{(L_\alpha(\nu),D)}_{0,\nu\lambda,\omega},E\big) \tag{27}$$

for all $E<0$. It follows from (22) that for every $\delta>0$, there exists $\alpha_0^->0$ (depending on $\delta,\nu,\lambda,\omega$ and E) such that,

$$n(H_{\alpha,\lambda,\omega},E)\geq [L_\alpha(\nu)]^d[N_{\nu\lambda}(E)-\delta] \tag{28}$$

for all $\alpha\in]0,\alpha_0^-[$.

To obtain the upper bound in (24), we set $\ell_\alpha(E):=2|\lambda U_0/E|^{1/\alpha}$ for $\alpha>0$, $E<0$ and argue that for every $\lambda>0$ and every $\omega\in\Omega_0$, the Neumann restriction of $H_{\alpha,\lambda,\omega}$ to $\mathbb{R}^d\backslash\Lambda_{\ell_\alpha(E)}$ cannot have any spectrum in $]-\infty,E]$. This implies the first inequality in

$$n(H_{\alpha,\lambda,\omega},E)\leq n\big(H^{(\ell_\alpha(E),N)}_{\alpha,\lambda,\omega},E\big)\leq n\big(H^{(\ell_\alpha(E),N)}_{0,\lambda,\omega},E\big), \tag{29}$$

the second one follows from monotonicity. Combining (29) and (22), we conclude that for every $\delta>0$ there exists $\alpha_0^+>0$ (depending on δ,λ,ω and E) such that

$$n(H_{\alpha,\lambda,\omega},E)\leq [\ell_\alpha(E)]^d\,[N_\lambda(E)+\delta] \tag{30}$$

holds for all $\alpha\in]0,\alpha_0^+[$. Equations (28) and (30) prove (24).

As to the validity of (6), we remark that the condition $E_0(\nu\lambda)<E$, which enters through $\nu_0(\lambda,E)$, guarantees the positivity of the lower bound in (24) for δ small enough. □

4. Persistence of Anderson localization

In this section we prove part (3) of Theorem 1 by a multiscale argument. The fractional-moment method [AENSS] should work as well, provided the single-site potential u satisfies the covering condition $u\geq v_0\chi_{\Lambda_1}>0$.

The multiscale analysis deals with restrictions of $H_{\alpha,\lambda,\omega}$ to finite volumes. These "finite volume operators" are required to have discrete spectrum in the range of energies we are interested in. Since we work with energies below the spectrum of the Laplacian, we choose

$$H_{\alpha,\lambda,\omega}(\Lambda_L(x)):=-\Delta-\lambda\gamma_\alpha\sum_{i\in\Lambda_L(x)}\omega_i u(\,\cdot-i)=:-\Delta+V_{\Lambda_L(x)} \tag{31}$$

(acting in $\mathrm{L}^2(\mathbb{R}^d)$) for the restriction of the operator $H_{\alpha,\lambda,\omega}$ to $\Lambda_L(x)$, the cube with edges of length L centered at $x\in\mathbb{R}^d$.

A crucial ingredient for the multiscale analysis is a Wegner estimate. In the non-ergodic situation we are facing here, we need it *uniformly in the location of the center of the box.*

Lemma 1. (Wegner estimate) *Assume that the single-site distribution* $\mathbb{P}(\omega_0\in\cdot\,)$ *has a bounded Lebesgue density* h. *Fix* $E'<0$ *and* $\lambda>0$. *Then, for any* $s\in]0,1[$, *there is a constant* $0\le Q_s=Q_s(\lambda,u,E')<\infty$ *such that for all* $\alpha\ge 0$, *all energies* $E\le E'$, *all lengths* $L>0$ *and all* $\eta\le|E'|/4$, *one has*

$$\sup_{x\in\mathbb{Z}^d}\mathbb{E}[\mathrm{tr}(P_{\Lambda_L(x)}(J_\eta))]\le Q_s\eta^s L^d, \tag{32}$$

where $J_\eta:=[E-\eta,E+\eta]$ *and* $P_{\Lambda_L(x)}(J_\eta):=\chi_{J_\eta}(H_{\alpha,\lambda,\omega}(\Lambda_L(x)))$ *is the spectral projection of* $H_{\alpha,\lambda,\omega}(\Lambda_L(x))$ *associated with the interval* J_η. *As a consequence,*

$$\sup_{x\in\mathbb{Z}^d}\mathbb{P}[\mathrm{dist}(\sigma(h_{\alpha,\lambda,\omega}(\Lambda_L(x))),E)\le\eta]\le Q_s\eta^s L^d. \tag{33}$$

Remark 9. If the single-site potential covers the unit cube, then [CoH] applies and one gets (32) with $s=1$ (see also [CoHK2] for a recent development).

Remark 10. One might have expected a volume correction in (32) due to the geometry of the potential. That this is not the case relates to the fact that we consider only energies below the spectrum of $-\Delta$. The decaying envelope makes it even harder to get an eigenvalue close to a given $E\le E'<0$. Actually, if the box $\Lambda_L(x)$ is far enough away from the origin, i.e. if

$$\inf_{y\in\Lambda_L(x)}\langle y\rangle\ge\left(\frac{2\lambda u_0}{|E'|}\right)^{1/\alpha}, \tag{34}$$

then $H_{\alpha,\lambda,\omega}(\Lambda_L(x))\ge\frac{1}{2}E'$ and $\mathbb{E}[\mathrm{tr}(P_{\Lambda_L(x)}(J_\eta))]=0$.

Proof of Lemma 1. We follow the strategy of [CoHN] and [CoHK1]. For convenience, let us write $d_0:=|E'|$ and $R_0(E):=(-\Delta-E)^{-1}$. Since $\mathrm{dist}(E,\sigma(-\Delta))\ge d_0$, one has

$$\begin{aligned}\mathrm{tr}(P_{\Lambda_L(x)}(J_\eta)) &= \mathrm{tr}(P_{\Lambda_L(x)}(J_\eta)(H_{\alpha,\lambda,\omega}(\Lambda_L(x))-E)P_{\Lambda_L(x)}(J_\eta)R_0(E))\\ &\quad-\mathrm{tr}(P_{\Lambda_L(x)}(J_\eta)V_{\Lambda_L(x)}R_0(E))\\ &\le\frac{\eta}{d_0}\,\mathrm{tr}(P_{\Lambda_L(x)}(J_\eta))-\mathrm{tr}(P_{\Lambda_L(x)}(J_\eta)V_{\Lambda_L(x)}R_0(E)).\end{aligned} \tag{35}$$

But notice that, using Cauchy–Schwarz's inequality and $\|R_0(E)\|\le d_0^{-1}$,

$$\begin{aligned} &|\operatorname{tr}(P_{\Lambda_L(x)}(J_\eta)V_{\Lambda_L(x)}R_0(E))| \\ &\qquad\le \frac{1}{d_0}\|P_{\Lambda_L(x)}(J_\eta)V_{\Lambda_L(x)}\|_2\|P_{\Lambda_L(x)}(J_\eta)\|_2 \\ &\qquad\le \frac{1}{2d_0^2}\operatorname{tr}(P_{\Lambda_L(x)}(J_\eta)V_{\Lambda_L(x)}^2)+\frac{1}{2}\operatorname{tr}(P_{\Lambda_L(x)}(J_\eta)). \end{aligned} \tag{36}$$

Since we took $\eta\le d_0/4$, (35) and (36) combine to give

$$\operatorname{tr}(P_{\Lambda_L(x)}(J_\eta))\le\frac{2}{d_0^2}\operatorname{tr}(P_{\Lambda_L(x)}(J_\eta)V_{\Lambda_L(x)}^2)\le\frac{2\lambda U_0}{d_0^2}\operatorname{tr}(P_{\Lambda_L(x)}(J_\eta)\widetilde{V}_{\Lambda_L(x)}), \tag{37}$$

where $\widetilde{V}_{\Lambda_L(x)}:=\lambda\gamma_\alpha\sum_{i\in\Lambda_L(x)}u(\,\cdot\,-i)\ge 0$ and the constant U_0 was defined in part (2) of Theorem 1. The usual (but crucial) observation is that

$$\widetilde{V}_{\Lambda_L(x)}=-\sum_{i\in\Lambda_L(x)}\frac{\partial V_{\omega,\Lambda_L(x)}}{\partial\omega_i}. \tag{38}$$

Next we pick a continuously differentiable, monotone decreasing function $f_\eta\colon\mathbb{R}\to[0,1]$ such that $f_\eta(\xi)=1$ for $\xi\le E-2\eta$ and $f_\eta(\xi)=0$ for $\xi\ge E+2\eta$. In particular, this function can be chosen such that $\chi_{J_\eta}\le -C\eta f_\eta'$ holds with some constant C, which is independent of η. It follows that (recalling also $\widetilde{V}_{\Lambda_L(x)}\ge 0$)

$$\begin{aligned} &\mathbb{E}[\operatorname{tr}(P_{\Lambda_L(x)}(J_\eta)\widetilde{V}_{\Lambda_L(x)})] \\ &\le C\eta\sum_{i\in\Lambda_L(x)}\mathbb{E}\left[\operatorname{tr}\left(f_\eta'(H_{\alpha,\lambda,\omega}(\Lambda_L(x)))\frac{\partial V_{\omega,\Lambda_L(x)}}{\partial\omega_i}\right)\right] &(39)\\ &\le C\eta\sum_{i\in\Lambda_L(x)}\mathbb{E}_{\omega_i^\perp}\int_0^1 h(\omega_i)\frac{\partial}{\partial\omega_i}\operatorname{tr}(f_\eta(H_{\alpha,\lambda,\omega}(\Lambda_L(x))))\,d\omega_i &(40)\\ &\le C\eta\,\|h\|_\infty\sum_{i\in\Lambda_L(x)}\mathbb{E}_{\omega_i^\perp}\operatorname{tr}\big(f_\eta\big(H_{\alpha,\lambda,\omega}^{(\omega_i=1)}(\Lambda_L(x))\big)-f_\eta\big(H_{\alpha,\lambda,\omega}^{(\omega_i=0)}(\Lambda_L(x))\big)\big). &(41) \end{aligned}$$

The average $\mathbb{E}_{\omega_i^\perp}$ in the above inequalities is over all random variables except ω_i. Now, using the spectral shift function, it follows from [CoHN] (or see [CoHK1, equations (A.12)–(A.14)]) that, for any $s\in]0,1[$,

$$\operatorname{tr}\big(f_\eta\big(H_{\alpha,\lambda,\omega}^{(\omega_i=1)}(\Lambda_L(x))\big)-f_\eta\big(H_{\alpha,\lambda,\omega}^{(\omega_i=0)}(\Lambda_L(x))\big)\big)\le C'(\lambda u_0)^{1-s}\eta^{s-1}, \tag{42}$$

uniformly for all $\alpha\ge 0$. The bound (32) follows from (37), (41), and (42). □

We are now ready to prove part (3) of Theorem 1.

Proof of part (3) *of Theorem* 1. Since boxes are independent at a distance, and we have the Wegner estimate of Lemma 1, it suffices to prove an initial-scale estimate. Again, this has to be done uniformly in the location of the center of the box, because the model lacks translation invariance. Then the bootstrap multiscale analysis of [GK1] applies. From now on, we fix $\lambda>0$.

The most common method to prove the initial-scale estimate consists in emptying the spectrum of the finite-volume operator $H_{\alpha,\lambda,\omega}(\Lambda_{L_0}(x))$ in an appropriately chosen interval $I_\lambda=[E_0(\lambda),E_1(\lambda)]$ at the bottom of the spectrum, see e.g., [CoH], [Klo1], [Klo3], [GK2] and [GK3]. There one can find proofs that in the ergodic situation $\alpha=0$ and for L_0 large enough, one gets $\sigma(H_{0,\lambda,\omega}(\Lambda_{L_0}(x)))\subset[E_1(\lambda)+m_0,+\infty[$ for some $m_0>0$ and all ω in some set of sufficiently large probability. Adding the envelope will only lift the spectrum up, and thus $\sigma(H_{\alpha,\lambda,\omega}(\Lambda_{L_0}(x)))\subset[E_1(\lambda)+m_0,+\infty[$ holds uniformly in $\alpha\geq0$ and $x\in\mathbb{R}^d$ with sufficiently large probability (independently of α). The Combes–Thomas estimate then provides the needed decay on the resolvent.

Having the initial-scale estimate and the Wegner estimate at hand, the bootstrap multiscale analysis of [GK1] can be performed. This provides for $\mathbb{P}$-a.e. ω the exponential decay of the eigenfunctions given in (7). By a center of localization of an eigenfunction $\varphi_{n,\alpha,\lambda,\omega}$ with energy $E<0$, we mean a point $x_{n,\alpha,\lambda,\omega}\in\mathbb{Z}^d$ such that $\|\chi_{x_{n,\alpha,\lambda,\omega}}\varphi_{n,\alpha,\lambda,\omega}\|=\sup_{x\in\mathbb{Z}^d}\|\chi_x\varphi_{n,\alpha,\lambda,\omega}\|$. To determine the location of such centers of localization, we proceed as follows. Set $L_E:=\max\{1,(2\lambda u_0|E|^{-1})^{1/\alpha}\}$. Assume that $|x_{n,\alpha,\lambda,\omega}|\geq(N+1)L_E$, with $N\geq1$, and consider the box $\Lambda_{NL_E}(x_{n,\alpha,\lambda,\omega})$. The spectrum of $H_{\alpha,\lambda,\omega}(\Lambda_{L_E}(x_{n,\alpha,\lambda,\omega}))$ is separated from E by a gap of size at least $|E|/2$. We estimate $\|\chi_{x_{n,\alpha,\lambda,\omega}}\varphi_{n,\alpha,\lambda,\omega}\|$ by the resolvent of $H_{\alpha,\lambda,\omega}(\Lambda_{L_E}(x_{n,\alpha,\lambda,\omega}))$. In the terminology of [GK1], this is called (EDI). We use the fact that $x_{n,\alpha,\lambda,\omega}$ maximizes $\|\chi_x\varphi_{n,\alpha,\lambda,\omega}\|$ and that the finite volume resolvent decays exponentially as $\exp(-NL_E|E|)$, by a Combes–Thomas type argument. This leads to an inequality of the form $1\lesssim\exp(-NL_E|E|)$, and thus to a contradiction if N is large enough. It remains to estimate N, and then we get that $|x_{n,\alpha,\lambda,\omega}|\leq(N+1)L_E$.

If $L_E=(2\lambda u_0|E|^{-1})^{1/\alpha}>1$ (which is only possible if $|E|<2\lambda u_0$ and $\alpha\leq1$), then $NL_E|E|=2\lambda u_0N(2\lambda u_0|E|^{-1})^{(1/\alpha)-1}$, which goes to infinity as $\alpha\downarrow0$. It is thus enough to take $N=(2\lambda u_0)^{-1}C$ with a large enough universal constant C. This implies that $|x_{n,\alpha,\lambda,\omega}|\lesssim(2\lambda u_0)^{(1/\alpha)-1}|E|^{-1/\alpha}$. If $L_E=1$, then $NL_E|E|=N|E|$, and we require $N=C|E|^{-1}$, with a large enough universal constant C. In this case $|x_{n,\alpha,\lambda,\omega}|\lesssim|E|^{-1}$. □

Acknowledgements. F.G. thanks the hospitality of UC Irvine where this work was done. F.G. also thanks P. Hislop for enjoyable discussions.

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Received April 18, 2006